

ON THE EFFECTS OF STOKES, RICHARDSON, AND STABILITY NUMBERS IN PERSISTENT AND ACCELERATING VORTICES

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1. Introduction

1.1 PERSISTENT VORTICES

Based on observations in stratified flow, Cotel [1,2] proposed a new model for stratified entrainment, in which a new parameter was asserted to be necessary to explain the observations. When a vortex is near an interface, the fluxes across the interface are determined by conventional parameters such as Richardson and Reynolds numbers as well as a new "persistence parameter." A measure of the stationarity of the vortex, the persistence parameter is essentially the ratio of the rotational to the translational speed of the vortex with respect to the interface. It plays as large a roll in stratified entrainment as any other parameter.

According to the theory, the fluxes across the interface would be reduced to their laminar values, independent of the fine-scale structure of the turbulence, if the interface is sufficiently stratified to remain essentially flat, $Ri > Re^{1/4}$. If a vortex can not engulf across the interface, all fluxes there must be purely diffusive. This is also true if the interface is a solid wall. In recent water tunnel experiments [3], vortex generators added strong, streamwise vortices to the flow at the upstream end of a corrugated plate. When each streamwise groove contained a single vortex in the optimum location, the vortices were stabilized. The wall heat flux measured in the bottom of a groove was reduced to a laminar value when the vortices were stable.

It is counter-intuitive that adding strong vortices to the flow would reduce the wall flux. It is also surprising that a turbulent flow would exhibit a laminar wall flux. There are two different kinds of turbulence in wall flows, the ordinary nonpersistent flow with relatively high, "turbulent" wall fluxes and the persistent case with low, "laminar" wall fluxes. With such unusual flux behavior, a natural question is the effect of persistence on two-phase flow.

1.2 ACCELERATING VORTICES

Self-similarity is one of the few powerful tools in turbulence research. Early on, Richardson and Obukhoff used the concept to deduce the behavior of the energy cascade in the inertial range of eddies in turbulence [4,5]. Because there is no distinguishable eddy size in this range, each eddy must behave like any other, except for a stretching in length and time scales. While self-similarity in space has received considerable attention, self-similarity in time has been relatively neglected.

Consider ordinary unforced turbulence at high Reynolds number. Viscosity sets the size of the smallest eddy, given by the Kolmogorov microscale λ_0 [6,7]. The ratio of the smallest to the largest eddy is

$$\lambda_0/\delta = Re_s^{-3/4}, \quad (1)$$

where δ is the largest eddy size and

$$\text{Re}_\delta = \Delta U \delta / \nu \quad (2)$$

is the Reynolds number of the largest eddies, whose characteristic velocity is ΔU . At large Re_δ , the microscale is much smaller than the large scale vortices. According to Richardson and Obukhoff, energy cascades from the largest eddies down towards the smallest in an essentially inviscid process without dissipation, until the Kolmogorov microscale is reached. The eddy Reynolds number at this point is

$$\text{Re}_{\lambda_0} \equiv \frac{v_{\lambda_0} \lambda_0}{\nu} \equiv 1, \quad (3)$$

where v_{λ_0} is the characteristic velocity of scale λ_0 . The kinetic energy is rapidly converted to thermal energy by viscous stresses.

At large Re_δ , there is a wide range of eddies larger than λ_0 but smaller than δ . Kinetic energy is transferred from one eddy to the next by inertial forces without appreciable energy loss in the inertial subrange. The kinetic energy flux, e , is constant so that

$$e = v_\lambda^3 / \lambda = \Delta U^3 / \delta. \quad (4)$$

Within the inertial subrange, an eddy of size λ and speed v_λ presumably hands over a large fraction of its energy in one turnover time given by

$$\tau_\lambda = \lambda / v_\lambda = \lambda^{2/3} / e^{1/3} = (\lambda / \delta)^{2/3} \tau_\delta, \quad (5)$$

where $\tau_\delta = \delta / \Delta U$ is the large scale vortex rotation period. A different question is how long it takes to reach scale λ from the start of the cascade. It has been suggested that self-similarity requires

$$dt = \text{const.} \lambda^{2/3} d\lambda / (e^{1/3} \lambda), \quad (6)$$

where dt is the differential time interval during which energy is transferred from scale λ to $\lambda - d\lambda$ [5]. Integration of (6) and the use of (4) provides the elapsed time $t(\lambda)$ to reach scale λ starting from scale δ or

$$t(\lambda) = [1 - (\lambda / \delta)^{2/3}] \tau_\delta. \quad (7)$$

By combining equations (5) and (7),

$$= 1 - t(\lambda) / \tau_\delta. \quad (8) \quad \tau_\lambda / \tau_\delta$$

The rotation period at scale λ is a linearly decreasing function of the elapsed time since the start of the cascade. This seems reasonable since there is no other time scale available. The radius of curvature of the function is therefore infinite, and therefore the relationship must be linear. The initial and final conditions of the energy cascade require the function to decline linearly in t .

2. Super-exponential forced turbulence

A new class of forced turbulence has recently been proposed [9]. The forcing is of form $\exp(t/\tau_v)$ where the e-folding time imposed on the vortices $\tau_v = \tau_v(t)$ is itself a function of time. Self-similar flow implies that there is no distinguished scale, either spatial or temporal. Hence the vortex rotation period $\tau_v(t)$ at any time t is always a constant fraction of the next rotation period, $\tau_v(t + \tau_v(t))$. Every rotation looks like every other one, except for a constant proportional change in scale. Expressed another way, the decrease in rotation period per rotation is a constant, independent of time, namely

$$[\tau_v(t) - \tau_v(t + \tau_v(t))] / \tau_v(t) = \beta, \quad (9)$$

where β is a constant. Suppose $\tau_v(t)$ is a general polynomial of order n given by

$$\tau_v(t) = \tau_v(0) + a_1 t + a_2 t^2 + \dots + a_n t^n. \quad (10)$$

After substituting equation (10) into equation (9) and some algebra, it is found that

$$[-a_1(\tau_0 + a_1 t + a_2 t^2 + \dots + a_n t^n) + O(t^{n^2})]/(\tau_0 + a_1 t + a_2 t^2 + \dots + a_n t^n) = \beta. \quad (11)$$

The left hand side has terms of order t raised to the (n^2-n) power while the right-hand-side is independent of t . This can only be satisfied if $n^2-n=0$. The solutions are $n = 0$ or $n = 1$.

Thus $a_i = 0$ for all $i > 1$, and $a_1 = -\beta$. For this to be true, the only possible self-similar form is

$$\tau_v(t) = \tau_v(0) - \beta t, \quad (12)$$

where $\tau_v(0)$ is the rotation period at $t = 0$. In dimensionless form,

$$\tau_v(t)/\tau_v(0) = 1 - \beta t/\tau_v(0). \quad (13)$$

If all self-similar vortices follow equation (13), the self-similar vortices in the inertial subrange must also. From a comparison between equations (7) and (13), evidently $\beta = 1$ following the energy cascade in unforced turbulence. This suggests that dissipation always vanishes at $\beta = 1$.

Figure 1 illustrates in graphical form the self-similar evolution for the general case of $\beta > 0$. At the end of the first rotation ($n=1$), the rotation period is reduced by a constant factor $(1 - \beta)$. At the end of the second rotation ($n=2$), τ_v is again reduced by the same factor, and so on. At finite time $t/\tau_v(0) = 1/\beta$, τ_v vanishes.

To an observer following the cascading energy in the inertial subrange, $\beta = 1$. However, to an observer following the large scale vortices in the same unforced flow, $\beta = -1$ [6,7]. The rotation period is approximately equal to the vortex chronological age, since there is no other time scale available. The value of β thus depends on the observer. There seem to be two different canonical coordinate frames, in analogy with Eulerian and Lagrangian viewpoints of fluid motion, that yield different values of β for the same flow. An observer may either remain at a fixed eddy scale or follow the energy. In ordinary, unforced turbulence, the largest eddies extract energy from the mean flow via instabilities and hand it off to the top of the cascade in the inertial subrange. Energy flows through the largest eddies. An observer remaining at the largest vortices could not follow the energy in unforced turbulence.

The reference frame of the observer can be defined in terms of the dimensionless flux of energy past him. Define γ to be the rate of energy flux e' past the observer normalized by the energy cascade flux e through the largest eddies so that

$$\gamma = e'/e = e'\delta/\Delta U^3. \quad (14)$$

For an observer fixed at scale $\lambda = \delta$, $\gamma = 1$ and $\beta = -1$ for unforced turbulence. For an observer moving down the cascade with the energy flux, $\gamma = 0$ and $\beta = 1$. For intermediate cases, a reasonable conjecture is a linear relation, where $\beta = 1 - 2\gamma$ or $\gamma = (1-\beta)/2$.

In the limit of $\beta = 1$, the implication is that the entrainment rate vanishes [9]. This implies that the cascade process is interrupted, shutting off the flow of energy from the largest eddies into the cascade. Such a situation is plausible in light of the fact that the hand-off process of energy requires something like a full rotation of the vortex. For $\beta = 1$, however, the time for a vortex to rotate or for a vortex sheet to roll up is just equal to the interval remaining before the singular time.

3. Vorticity

The self-similar evolution represented by equation (13) is also the solution to the scalar equation

$$D\omega/Dt = \beta\omega^2, \quad (14)$$

if $\omega = 1/\tau_v$. The vector vorticity equation for constant density flow in the absence of body forces is

$$D\boldsymbol{\omega}/Dt = (\nabla\mathbf{u}) \boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}. \quad (15)$$

This suggests that, for inviscid self-similar flow, the magnitude of the nonlinear vortex stretching term $(\nabla\mathbf{u}) \boldsymbol{\omega}$ corresponds to $\beta\omega^2$. Thus, there is a kind of symmetry between the rate of deformation tensor and the vorticity, such that their inner product is always proportional to the square of the magnitude of the vorticity. The rate of deformation becomes proportional to the vorticity magnitude, and the proportionality coefficient is simply β .

4. Related experiments

4.1 EXPONENTIAL JET

A flow with a constant imposed time scale is the exponential jet [11,12], which exhibits a constant vortex rotation period for the largest vortices. Figure 2 illustrates the evolution of the large-scale rotation period as a function of time. All known turbulent self-similar are represented as a straight line on this figure, differing only in their slope, $-\beta$. The exponential jet entrains and mixes at a somewhat lower rate than the unforced jet.

4.2 TRANSVERSE EXPONENTIAL JET

In an exponential transverse jet, vortices are formed in the near field with an initial rotation period controlled by the nozzle time, the ratio of local nozzle diameter to speed [13]. However, as the vortices advect downstream, they are subjected to subsequent nozzles that increase in size and speed in an exponential way. The corresponding e-folding time introduces a second time scale on top of the first one. The ratio of these two times is an acceleration parameter α , a measure of the forcing. As α increases above the value of one, vortex rollup and entrainment are reduced. There is less time available for the vortices to complete a rotation.

4.3 GENERALIZED RAYLEIGH-TAYLOR FLOW

In order to test the conjecture that entrainment should vanish in the limit of $\beta \rightarrow 1$ in super-exponential forcing, a generalized Rayleigh-Taylor flow was simulated in two-dimensional inviscid calculations using the CLAWPACK code [14,15]. The density ratio across the interface was two, and the reduced acceleration g' was given a super-exponential profile in time with an initial dimensionless value of 0.1. The initial dimensionless e-folding time was one. When β was negative or zero, the interface rapidly formed into a mushroom shape as baroclinic vortex sheets roll up into discrete vortices. However, when β is one, the vortex sheet does not roll up at all. This is anticipated for all initial disturbances larger than about $g_0'\tau_0^2$. A plot of the vortex size δ_v , normalized by the total layer thickness δ , as a function of β is shown in figure 3. Here, the ratio δ_v/δ was evaluated at fixed δ . As $\beta \rightarrow 1$, the ratio δ_v/δ decreases to zero. Vortex sheet rollup is then completely inhibited.

The chaotic mixing regions associated with entrainment in accelerating turbulence have recently been explored by Govindarajan [16]. The connection between the chaotic mixing regions and β has not yet been determined, but the prime suspect is the motion of the saddle point on the separatrix.

When $\beta = 1$, the vortex sheet does not roll up, even though the remaining time is equal to the instantaneous rotation period. Furthermore, there is some inhibiting effect on vortex growth even when $\beta < 1$. J. Jimenez (personal communication) has pointed out that this implies the flow must be stabilized by the acceleration, in contrast to merely not having sufficient available time for rollup. Future work will analyze the stability.

5. Two-phase flow

5.1 PERSISTENT VORTICES

Define the persistence parameter T to be the number of rotations a vortex makes before it moves a distance with respect to the nearby interface equal to its diameter. In the persistent limit of $T \gg 1$, we expect particle separation for Stokes numbers $\frac{1}{T} \leq St \leq 1$, since the vortex makes many rotations while stationary, like a centrifuge. This contrasts with ordinary, nonpersistent turbulence, where particle separation only occurs for $St \approx 1$ [17].

5.2 ACCELERATING VORTICES

For a self-similar, accelerating vortex, $\tau_v = 1 - \beta t$, so the Stokes number $St \equiv \frac{\tau_p}{\tau_v}$ increases

without limit as $t \rightarrow \frac{\tau_v}{\beta}$, where τ_p is the particle relaxation time. If the Stokes number is initially greater than one, it remains so for all time. The particles are always uncoupled from the fluid. However, if the Stokes number is initially less than one, it increases to unity at time $t = \frac{\tau_0}{\beta} \left(1 - \beta \frac{\tau_p}{\tau_0}\right)$. The particles are then slung out of the vortex as it spins up.

As an example of the stability parameter, consider a forced Rayleigh-Taylor flow, in which the acceleration goes as $g = \frac{g_0}{\left(1 - \beta \frac{t}{\tau_0}\right)^2}$, and the density contrast is solely due to dense

particles. The stability parameter is then $StRi = \frac{g \tau_p \tau_v}{\delta} = \frac{g_0 \tau_p \tau_0}{\delta} \left(1 - \beta \frac{t}{\tau_0}\right)$, where δ is the

scale of the vortex. The stability parameter decreases with time. So if $g_0 < \frac{\delta}{\tau_p \tau_0}$, then

$StRi < 1$ for all time, and the particles immediately fall out. On the other hand, if $g_0 > \frac{\delta}{\tau_p \tau_0}$,

then the particles do not fall out initially. Only at time $t = \frac{\tau_0}{\beta} \left(1 - \frac{\delta}{g_0 \tau_p \tau_0}\right)$, the stability

parameter becomes unity and the particles fall out.

However, depending on the value of β , the particles may first be slung out radially. If

$\beta > \frac{\delta}{g_0 \tau_p^2}$, the particles will reach unity Stokes number before unity stability parameter, and

they will be centrifuged out instead. The particle motion may thus influence the flow stability.

6. Conclusions

Two new classes of turbulent flows have been recently recognized. In the first, the large-scale vortices are stationary near an interface. If the interface is strongly stratified or a solid wall, the fluxes are laminar. A vortex in the persistent limit will centrifuge out particles over a wide range of the Stokes number, since the vortex is stable and completes many rotations in place. In the second, self-similar acceleration requires a vortex rotation period to decrease linearly with time. The entrainment and dissipation rates depend solely on the logarithmic rate of decrease in rotation period. If the latter is large, vortex rollup is strongly inhibited. For vortices in an accelerating Rayleigh-Taylor flow, particles will eventually fall out unless the rotation period decreases at a sufficient rate, in which case the particles will be slung out radially.

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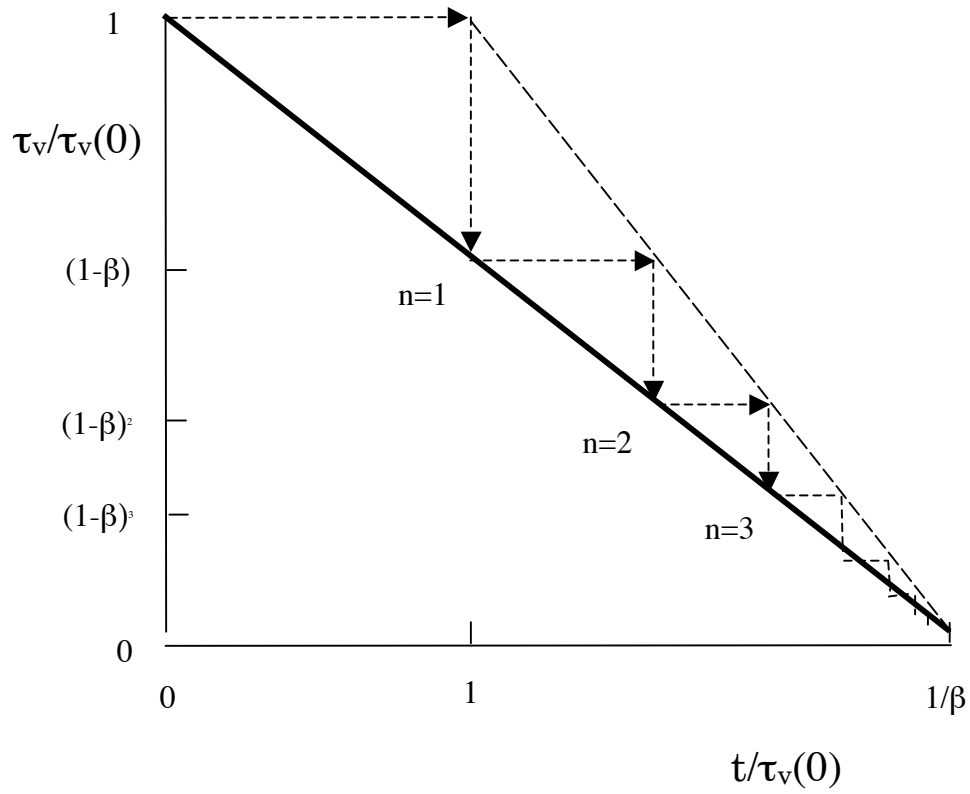


Fig. 1 Temporal evolution of the vortex rotation period τ_v for $0 < \beta < 1$ in self-similar flow.

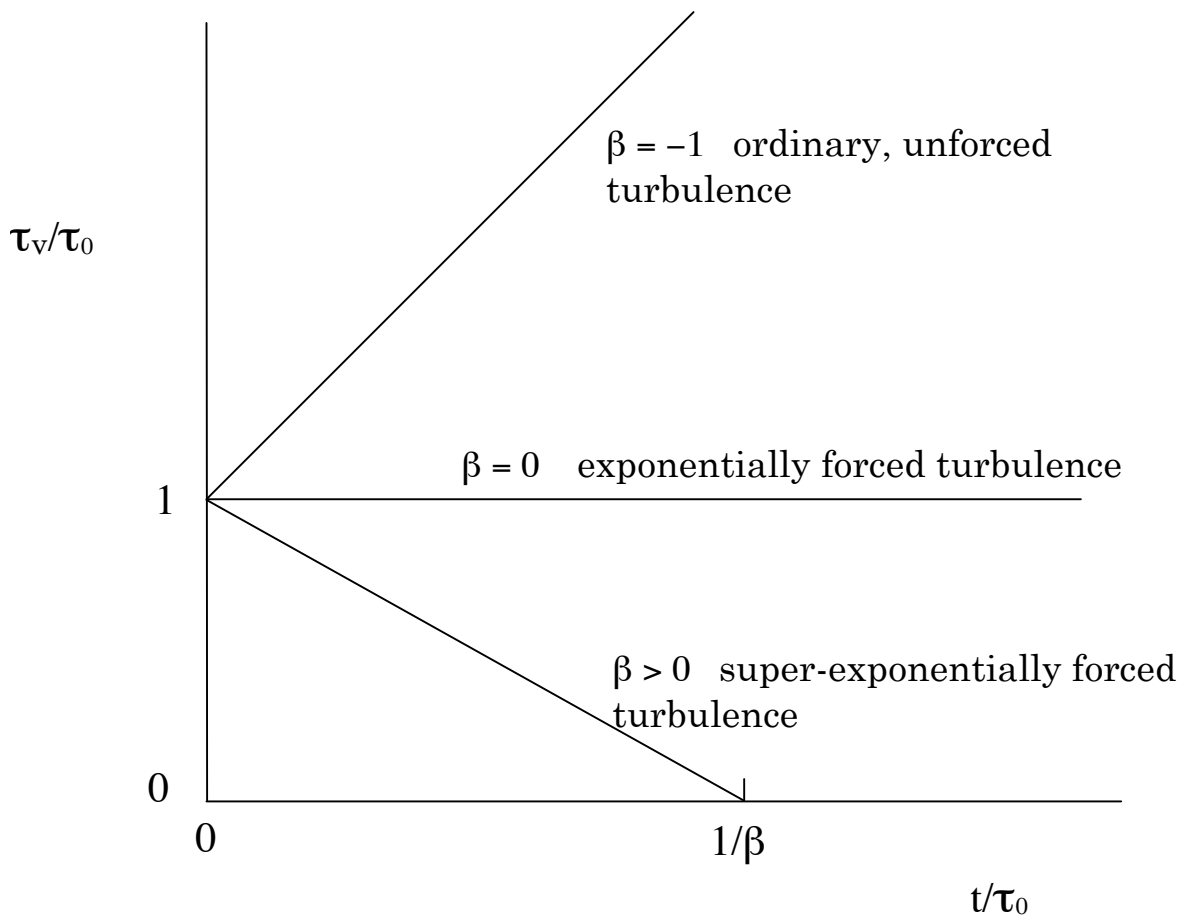


Fig. 2 Vortex rotation period $\tau_v(t)$ for all self-similar flows

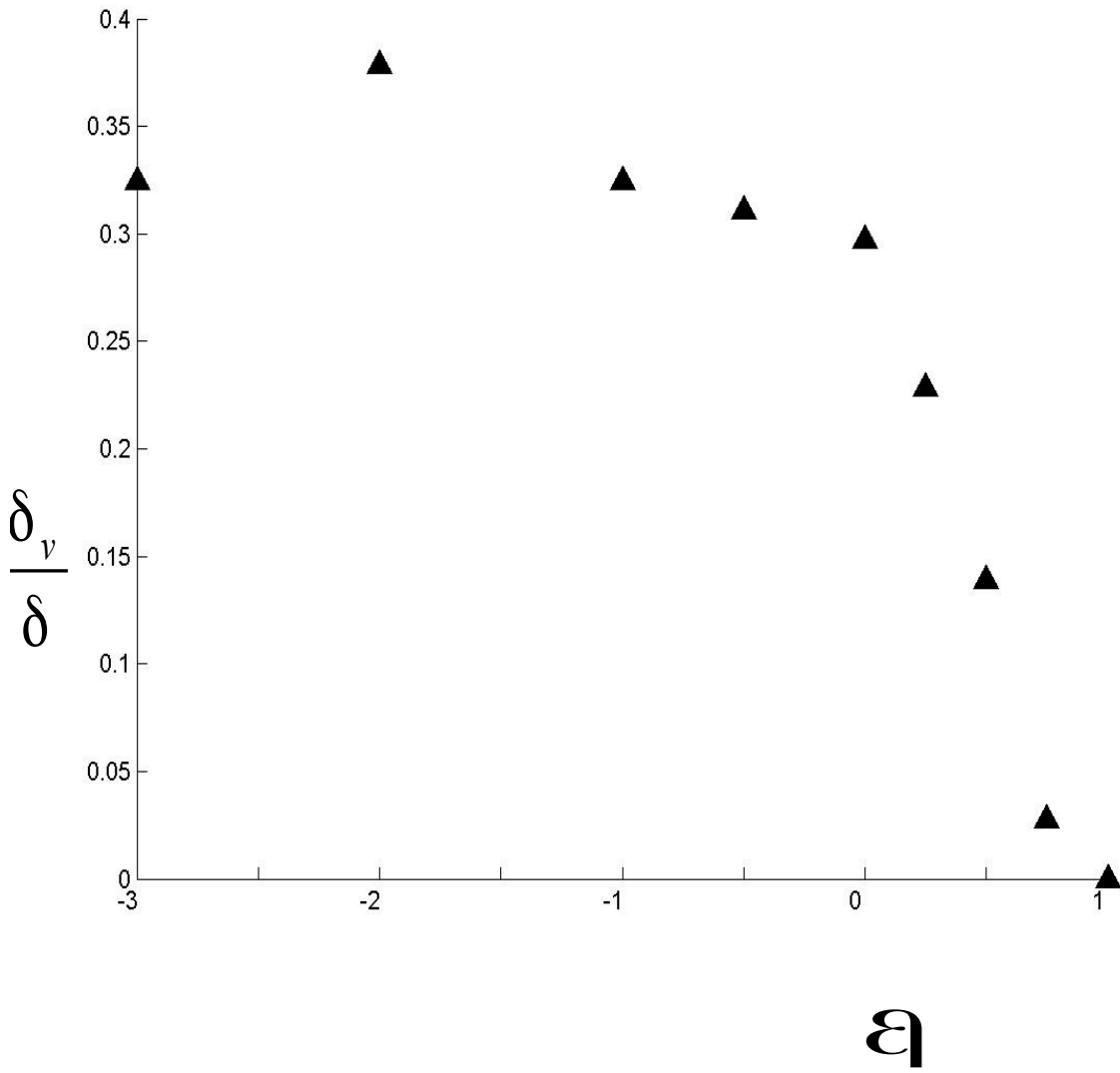


Fig. 3 Vortex size as a function of β .

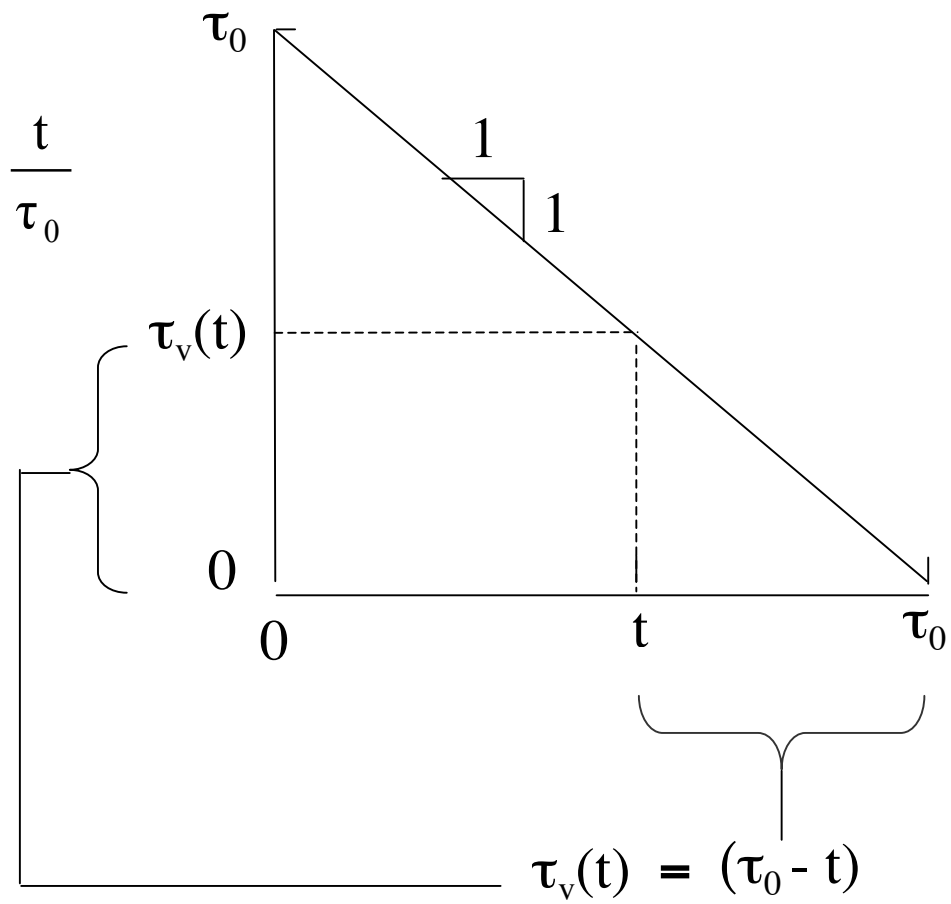


Fig. 4 At every instant, the current rotation period is just equal to the remaining available time for $\beta = 1$. The vortex sheet never rolls up.