Inter. Communications in Heat & Mass Transfer, vol. 16, no. 1, p. 89-97

# INFINITE PRANDTL NUMBER VARIABLE VISCOSITY FREE CONVECTION WITH SUCTION

George W. Bergantz
Department of Geological Sciences, AJ-20
The University of Washington, Seattle, Washington, 98195

## ABSTRACT

The equations describing variable viscosity infinite Prandtl number free convection from a vertical surface with suction were solved using the local nonsimilarity technique. Two heat transfer regimes are recognized: at low values of suction, variable viscosity dominates and can substantially reduce the heat transfer, at moderate to high suction the temperature dependent viscosity plays no role in the description of the heat transfer. Vertical velocities are reduced in magnitude and shifted away from the boundary as the temperature dependence of viscosity is increased.

## Introduction

Interest in the influence of suction on the heat and mass transfer associated with forced and free convection has traditionally been motivated by the influence of cross stream mass transfer on boundary layer stability and heat transfer augmentation. Previous work addressing the influence of suction on free convection includes Eichorn [1] who established the streamwise dependence of suction which will yield similarity solutions. Sparrow and Cess [2] employed a series solution for spatially uniform suction at the boundary. Merkin [3] treated the problem of constant suction and found a constant thickness boundary layer in the asymptotic limit. Both Brdlik and Michalov [4] and Parikh et al. [5] provide interferograms of free convection with suction, additional numerical cooboration of the heat transfer results were given in [5]. Minkowycz and Sparrow [6] investigate the problem of free convection with suction as a vehicle for their numerical approach to systems of coupled ordinary differential equations. In the context of suction induced by crystallization, Lapadula and Mueller [7] treat the effects of of suction as a linear correction to the usual natural convection heat transfer coefficient.

The application of these works to the geophysical setting, and in particular to the study of magma (molten rock), is frustrated in that magmas are high Prandtl number fluids with a strong dependence of viscosity on temperature: the viscosity can vary many orders of magnitude in the temperature ranges of interest [8], [9], [14]. With this setting in mind we here develop a quantitative treatment of infinite Prandtl number variable viscosity free convection from a vertical surface with suction. The pressure differences that drive suction are due to the effects of the moving boundaries attendant with the crystallization process; we will not specifically address those processes here.

## Analysis

Following the lead of Carey and Mollendorf [10] we give the variable viscosity boundary layer equations describing steady free convection from a vertical surface where we invoke the usual Boussinesq and boundary layer assumptions and the effects of suction:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho_o}\frac{d\mu}{dT}\frac{\partial T}{\partial x}\frac{\partial v}{\partial x} + \frac{\mu}{\rho_o}\frac{\partial^2 v}{\partial x^2} + g\beta(T_m - T) \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial x^2} \tag{3}$$

where variables and symbols are defined in the nomenclature and where the first term on the right hand side of (2) accounts for the temperature dependence of viscosity. The boundary conditions are

$$v(0,y) = 0, \quad u(0,y) = -u_w, \quad T(0,y) = T_w \tag{4}$$

$$\lim_{x \to \infty} \left\{ v(x,y) = 0, \quad T(x,y) = T_m \right\}$$

We next introduce a stream function, dimensionless temperature and the desired form of the temperature dependent viscosity function

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}, \quad \theta = \frac{T - T_w}{T_m - T_w} \tag{5}$$

$$\mu = \mu_o \exp\left[a\,(T_m - T)\right] \tag{6}$$

from which (1) - (3) yields

$$\frac{\partial \psi}{\partial x}\left(\frac{\partial^2 \psi}{\partial y \partial x}\right) - \frac{\partial \psi}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = A\nu_o \exp\left[A(1 - \theta)\right]\frac{\partial \theta}{\partial x}\frac{\partial^2 \psi}{\partial x^2}$$

$$- \nu_o \exp\left[A(1 - \theta)\right]\frac{\partial^3 \psi}{\partial x^3} + g\,\beta(T_m - T_w)\,(1 - \theta) \tag{7}$$

and

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \kappa \frac{\partial^2 \theta}{\partial x^2} \tag{8}$$

Next, we introduce the following dimensionless variables:

$$\psi = \kappa\, Ra_y^{1/4} f(\xi,\eta), \quad \eta = \frac{x}{y} Ra_y^{1/4}, \quad \xi = -\frac{u_w y}{\kappa\, Ra_y^{1/4}} \tag{9}$$

The presence of spatially constant suction at the wall precludes a direct similarity solution hence we invoke the local nonsimilarity technique of Sparrow and Yu [11]. The independent variable $\xi$ incorporates the streamwise dependence of the solution that arises due to the presence of suction. In the absence suction, $\xi$ is identically zero and the usual similarity variable obtain. Also note that the similarity variables are written in terms of a Rayleigh number rather than a Grashof number which is in keeping with the well posed arguments of Bejan [12].

The governing equations are now

$$\frac{3}{4\,Pr}\left(f_\eta^{\,2} - f\,f_{\eta\eta}\right) - \frac{\xi}{4\,Pr}\left(f_\xi\,f_{\eta\eta} - f_\eta\,f_{\xi\eta}\right) = \exp\left[A\,(1-\theta)\right]\left(A\theta_\eta\,f_{\eta\eta} - f_{\eta\eta\eta}\right) + (1-\theta) \quad (10)$$

$$\theta_{\eta\eta} - \frac{3}{4}f\,\theta_\eta + \frac{\xi}{4}\left(f_\eta\,\theta_\xi - f_\xi\,\theta_\eta\right) = 0 \quad (11)$$

The next step in the local nonsimilarity technique is to eliminate the explicit appearance of $\xi$ derivatives by defining additional variables [11]:

$$g\,(\eta,\xi) = f_\xi\,,\quad \phi\,(\eta,\xi) = \theta_\xi \quad (12),\ (13)$$

and substituting them into (10) - (11). These expressions are subsequently differentiated with respect to $\xi$ which yields two new equations for g and $\phi$. Terms are deleted from the resulting equation set that are involved with the operator ($\xi\ \partial/\partial\xi$). This yields

$$f_{\eta\eta\eta} = \left(-\frac{1}{\exp\left[A\,(1-\theta)\right]}\right)\left[\frac{3}{4\,Pr}\left(f_\eta^{\,2} - f\,f_{\eta\eta}\right) - \frac{\xi}{4\,Pr}\left(g\,f_{\eta\eta} - f_\eta\,g_\eta\right) - (1-\theta)\right] + A\theta_\eta\,f_{\eta\eta} \quad (14)$$

$$g_{\eta\eta\eta} = \left(-\frac{A\phi}{\exp\left[A\,(1-\theta)\right]}\right)\left[\frac{3}{4\,Pr}\left(f_\eta^{\,2} - f\,f_{\eta\eta}\right) - \frac{\xi}{4\,Pr}\left(g\,f_{\eta\eta} - f_\eta\,g_\eta\right) - (1-\theta)\right]$$
$$- \left(\frac{1}{\exp\left[A\,(1-\theta)\right]}\right)\left[\frac{3}{4\,Pr}\left(2f_\eta g_\eta - g f_{\eta\eta} - f g_{\eta\eta} + \phi\right)\right] + A\phi_\eta\,f_{\eta\eta} + A\theta_\eta\,g_{\eta\eta} \quad (15)$$

$$\theta_{\eta\eta} = \frac{3}{4}f\,\theta_\eta - \frac{\xi}{4}\left(f_\eta\,\phi - g\,\theta_\eta\right) \quad (16)$$

$$\phi_{\eta\eta} = \frac{3}{4}\left(g\,\theta_\eta + f\,\phi_\eta\right) - \frac{1}{4}\left(f_\eta\,\phi - g\,\theta_\eta\right) \quad (17)$$

thus the system (14) - (17) describes variable viscosity free convection with suction to the first level of truncation in the local nonsimilarity variable technique.

We seek further simplification by invoking the high Prandtl number character of magmas: kinematic viscosities can typically be 10 to 500 $m^2$ / s, thermal diffusivities of the order $10^{-6}$ $m^2$ / s. These values yield large Prandtl numbers which gives rise to a much thicker velocity boundary layer relative to the thermal boundary layer. This translates into a flow regime that can be partitioned in the following way: an inner region where the momentum balance is between buoyancy and viscous forces and where the velocity maximum occurs and an adjacent outer region where the balance is between inertia and viscosity. This structure has been formally addressed by Kuiken [13] who employed matched asymptotic expansions to investigate this dual layer structure and also by Morris [15]. In the infinite Prandtl number limit, (14) - (15) reduce to

$$f_{\eta\eta\eta} = \frac{(1-\theta)}{\exp\left[A(1-\theta)\right]} + A\theta_\eta f_{\eta\eta} \tag{18}$$

$$g_{\eta\eta\eta} = \frac{\phi\,[A(1-\theta)-1]}{\exp\left[A(1-\theta)\right]} + A\phi_\eta f_{\eta\eta} + A\theta_\eta g_{\eta\eta} \tag{19}$$

where we note that there are no changes in the heat equations (16) - (17). The boundary conditions for the system (16) - (19) are:

$$\begin{gathered} \eta = 0:\ f_\eta = 0,\ g_\eta = 0,\ \phi = 0,\ \theta = 0,\ g = -1,\ f = \xi \\ \lim_{\eta\to\infty} \left\{ f_{\eta\eta} \to 0,\ g_{\eta\eta} \to 0,\ \phi \to 0,\ \theta \to 1 \right\} \end{gathered} \tag{20}$$

Note that the boundary condition for the vertical velocity is not zero velocity at infinity but rather zero shear stress which is the appropriate condition for the assumptions that yielded the reduced set of equations (18) - (19). This is discussed by Kuiken [13] (see equation (43) therein) and also in some length in [9]. It was found that the form of this condition really makes little difference in the numerical value of the heat transfer coefficient: both zero shear stress and zero velocity boundary conditions were invoked and the numerical values of the Nusselt number differed only a few percent at most.

This set of equations was subsequently solved by employing the algorithm SUPORQ which is the nonlinear version of the SUPORT code written by Scott and Watts [16]. Briefly summarized this code employs quasilinearization and superposition coupled with orthonormalization and a variable step Runge-Kutta-Fehlberg scheme. For the results given below, the boundary conditions we satisfied to $10^{-6}$.

Of immeadiate interest is the manner in which the competing effects of suction, which tends to thin the thermal boundary layer, and variable viscosity which tends to push the fluid motion away from the wall, manifest themselves in the heat transfer coefficient. The Nusselt number can be given as

$$\frac{Nu_y}{Ra_y^{1/4}} = \left.\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} \tag{21}$$

and is shown in Figure 1 for differing values of the viscosity parameter A. At low values of $\xi$, the effects of variable viscosity dominate the heat transfer which can be reduced by as much as one half by the influence of variable viscosity. For higher values of $\xi$, the suction effects dominate the variable viscosity effects: the curves quickly become coincident with the A equals zero curve and the heat transfer is given by

$$\frac{Nu_y}{Ra_y^{1/4}} = -\xi \tag{22}$$

This behavior suggests that the fluid flow and heat transfer regimes are becoming decoupled in the high suction limit: the boundary layer thickness no longer increases in the streamwise direction, the *net* enthalpy advected into any control volume becomes vanishingly small and hence in the high suction limit the buoyancy plays no role in description of the heat transfer. This kind of behavior has been discussed previously [1], [3], [5]. What is of interest here is that the curves converge for relatively moderate amounts of suction and hence the effects of variable viscosity on the heat transfer can be ignored for $\xi$ greater than about 1.5.

The influence of variable viscosity on the vertical velocity field is shown in Figure 2. As the influence of temperature on viscosity (A) increases, the flow field moves away from the wall and decreases in magnitude. The fluid is thus minimizing shear stresses by moving the regions of higher velocity out into the effectively isoviscous portion of the fluid [14], [15]. This result was not influenced to any degree by the form of the velocity boundary condition in the farfield. In addition, it was found that the velocity maximum, where the shear stresses equal zero, was nearly coincident with the edge of the thermal boundary layer.

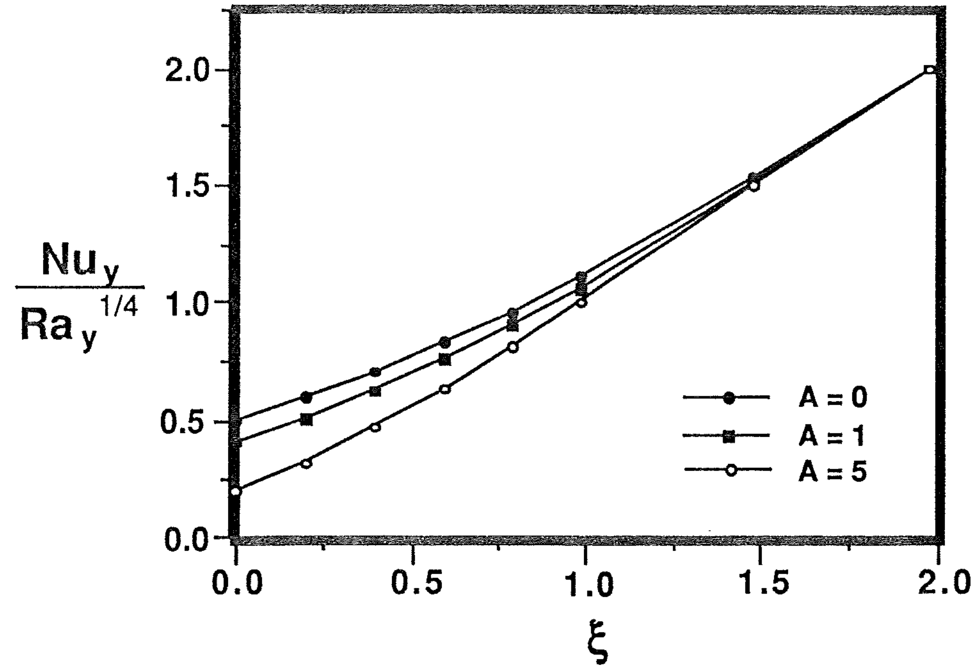


Figure 1. Nusselt-Rayleigh number relationship plotted as a function of $\xi$, the dimensionless suction variable, for differing values of A.

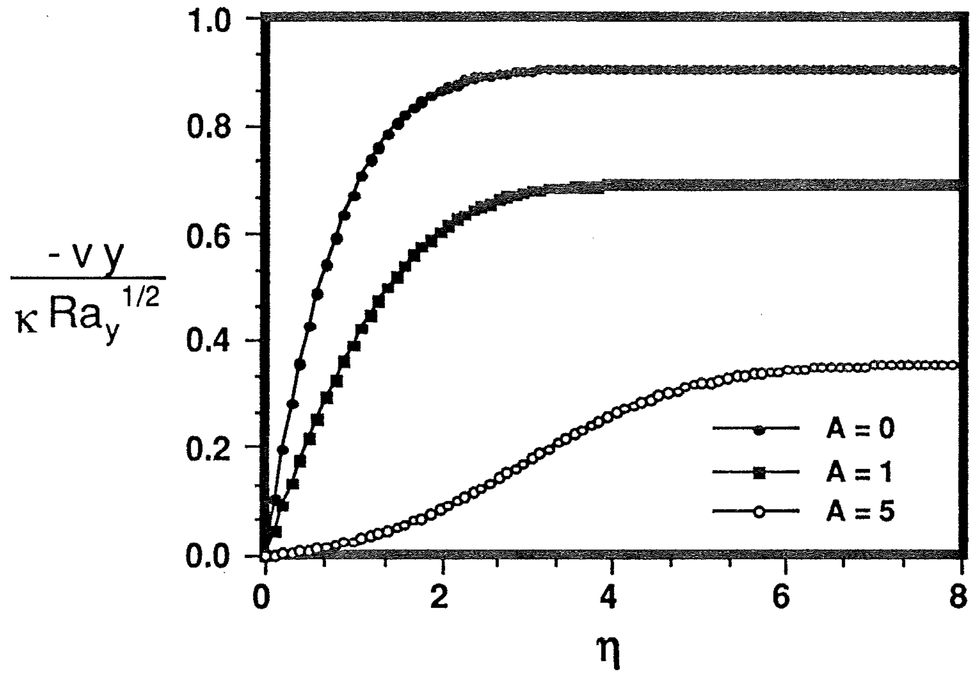


Figure 2. Dimensionless vertical velocity as a function of the similarity variable $\eta$ for differing values of the variable viscosity coefficient, $\xi$ is .25.

## Conclusions

The equations describing variable viscosity free convection for a infinite Prandtl number fluid from a vertical surface with suction have been solved. For low values of suction, the presence of temperature dependent variable viscosity can reduce the heat transfer by as much as a half. At moderate to high values of suction the heat transfer becomes independent of the viscosity function suggesting the approach to the suction dominated limit.

The influence of variable viscosity is readily manifested in the magnitude and distribution of the vertical velocity: a strong dependence of viscosity on temperature shifted the flow field away from the wall with a concomitant decrease in the maximum amplitude.

## Nomenclature

| | |
|---|---|
| $a$ | constant in variable viscosity as given in (6) |
| $A$ | $a(T_m - T_w)$ |
| $f$ | dimensionless stream function defined in (9) |
| g | auxiliary nonsimilarity variable defined in (12) |
| $g$ | scalar acceleration due to gravity |
| Pr | Prandtl number ($= \nu / \kappa$) |
| $Ra_y$ | Rayleigh number ($= g\, \beta \Delta T y^3 / \nu\kappa$) |
| $T, T_m, T_w$ | temperature of the fluid, reservoir temperature, temperature at wall |
| $u, u_w$ | horizontal velocity, suction velocity |
| $v$ | vertical velocity |
| $x$ | horizontal coordinate |
| $y$ | vertical coordinate |

### Greek Symbols

| | |
|---|---|
| $\beta$ | volumetric coefficient of thermal expansion |
| $\kappa$ | molecular thermal diffusivity |
| $\eta$ | similarity variable defined in (9) |
| $\theta$ | dimensionless temperature defined in (5) |
| $\mu$ | dynamic viscosity |
| $\nu, \nu_o$ | kinematic viscosity, reference viscosity |
| $\xi$ | nonsimilarity variable defined in (9) |

| | |
|---|---|
| $\rho_o$ | reference density |
| $\phi$ | auxiliary nonsimilarity variable defined in (12) |
| $\psi$ | stream function in terms of x and y as defined in (5) |

## References


[1] R. Eichorn, J. Heat Transfer *82,* 260, (1960).

[2] E. M. Sparrow and Cess, J. Heat Transfer, *83*, 387, (1961).

[3] J. H. Merkin, Int. J. Heat Mass Transfer, *15*, 989, (1972).

[4] P. M. Brdlik and V. A. Mochalov, J. Engineering Physics, *10*, 1, (1966).

[5] P. G. Parikh, R. J. Moffat, W. M. Kays and D. Bershader, Int. J. Heat Mass Transfer, *17*, 1465, (1974).

[6] W. J. Minkowycz and E. M. Sparrow, Numer. Heat Transfer, *1*, 69, (1978).

[7] C. A. Lapadula and W. K. Mueller, Int. J. Heat Mass Transfer, *13*, 13, (1970).

[8] A. R. McBirney and T. Murase, Ann. Rev. Earth Planet. Sci., *12*, 337, (1984).

[9] F. J. Spera, D. A. Yuen and S. J. Kirschvink, J. Geophysical Res., *87*, 8755 (1982).

[10] V. P. Carey and J. C. Mollendorf, Proc. 6th. Int. Heat transfer Conf., Toronto, *2*, 211, (1978).

[11] E. M. Sparrow and H. S. Yu, J. Heat Transfer, *93*, 328, (1971).

[12] A. Bejan, *Convection Heat Transfer,* John Wiley & Sons, New York, (1984).

[13] H. K. Kuiken, J. Engineering Mathematics, *2*, 355, (1968).

[14] R. H. Nilson, A. R. McBirney and B. H. Baker, J. Volcan. Geothermal Res., *24*, 25, (1985).

[15] S. Morris, J. Fluid Mechanics, *124*, 1, (1982).